Deterministic Variability Analysis for Intermediate Storage in Noncontinuous Processes

Part I: Allowability Conditions

Noncontinuous processes involving batch and semicontinuous processing operations are typically subject to significant levels of process parameter variations. Intermediate storage is commonly used to mitigate the effects of these variations. A taxonomy and analysis is presented of the various types of parameter variations: elementary and composite, single and multiple, homogeneous and mixed, overlapping and nonoverlapping. Sufficient conditions are developed which insure that continuity of periodic operation can be maintained in the presence of these various types of process variations. These conditions serve as the basis of storage sizing procedures developed in the companion paper.

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SCOPE

Noncontinuous processing facilities involving batch and semicontinuous operations are usually subject to a higher degree of process variability than steady-state processes. This variability is introduced by operator vagaries, fluctuations in utility availabilities, minor equipment malfunctions, or recipe inaccuracies. Variations in basic process parameters such as transfer flow rates, batch sizes, or batch start times introduce delays of operation as stages either wait or are blocked due to the tardiness or busyness of other stages. Moreover, fluctuations arising at one point in a serial process do propagate along the chain, often growing in magnitude or initiating similar or different additional fluctuations. One of the roles of intermediate storage is to mitigate the effects of such variations on the essentially periodically operating process by preventing their propagation between processing stages.

The quantitative analysis of this role of intermediate storage has been termed variability analysis (Karimi and Reklaitis, 1984). As reviewed there, such analyses have taken two basic directions: stochastic variability analysis and deterministic variability analysis. The former approach seeks to evaluate the effects of random process variations on the production capacity given a specified level of intermediate storage and has been mainly used to study discrete parts manufacturing systems. Early applications of the stochastic approach to batch/semicontinuous processes were reported by Stover (1963) and Smith and Rudd (1964). The deterministic approach considers specific patterns or combinations of variations and seeks to eliminate their effects by providing appropriate levels of intermediate storage. Oi et al. (1979) adopted this approach to study a system involving L parallel batch units, an intermediate storage facility, and a single downstream continuous unit with either constant or variable processing rate. Conditions were derived which single as well as multiple, nonoverlapping variations in the schedule of the batch units must satisfy to maintain continuity of operation. In subsequent work, Oi (1982, 1983) extended the above analysis to simultaneous multiple revisions of schedule (start times of successive batches). Takamatsu et al. (1984) analyzed schedule revisions and batch size variations with constant batch unit cycle times for batch processes involving stages with identical parallel units and fixed, equal time intervals between the starting moments of the units in a given stage.

In this paper, a taxonomy and comprehensive analysis is presented of the different kinds of process parameter variations occurring in batch/semicontinuous processes. Using the structure and characteristics of the different categories of variations, we develop sufficient conditions which these variations must satisfy in order to maintain continuity of operation. The key assumption in the analysis is that the variations introduced at any given time are temporary and the process returns to its nominal parameter values after the variations have occurred. The process is thus periodic except for the short-term nonperiodicity introduced by the variations. We first consider single elementary variations and then extend the results to sets of general multiple variations. Although for uniformity in presentation the sufficient conditions are derived for the serial configuration consisting of a storage vessel intermediate to two batch/semicontinuous units, some of the results extend directly to the general case involving multiple parallel units in each stage. The sufficient conditions are conservative in that they are stated in terms of the permanent, residual effects on system operation resulting from the variations rather than on the transients which occur during the time of variation.

CONCLUSIONS AND SIGNIFICANCE

A taxonomy was presented of the various types of process parameter variations in noncontinuous processes and a study was made of their characteristics. The major classification of variations was into elementary and composite categories. A sufficient condition was developed which single elementary variations must satisfy in order to maintain continuity of the otherwise periodic operation of *L-M* processes. A similar condition or allowability theorem was derived for a set of multiple elementary variations with or without overlaps occurring in a serial process.

An important consequence of this theorem is that it establishes the superimposability of elementary variations. Composite variations were analyzed by decomposing them into elementary variations and exploiting the superimposability of elementary variations. Using these features, a general holdup function was developed for representing any general set of multiple variations in terms of elementary variations. This construction then provided the direct mechanism for characterizing the allowability of any general set of multiple variations.

This work thus establishes a basis for the understanding and quantitative analysis of the role of intermediate storage in mitigating the effects of process variations. The allowability conditions are a key step in the derivation of the storage sizing procedures described in Part II. Moreover, the conditions can also be used in controlling the continuity of operation. Thus the results presented are important in both the design and the operation of noncontinuous processes.

PROCESS MODEL

Assumptions

Consider the primitive process shown in Figure 1. Following the terminology introduced in Karimi and Reklaitis (1984), we define an *L-M* or parallel process to be one in which there are L batch/semicontinuous units in the upstream stage, M batch/semicontinuous units in the downstream stage, and both stages are separated by a storage vessel of size V^*. In this notation, a serial process is denoted as a 1-1 process. For purposes of this paper the *L-M* process will be characterized by the nominal or average values of its parameters. For instance, V_i represents the nominal value of the batch size of ith unit, while ω_i represents the nominal value of the cycle time of ith unit. The basic assumptions regarding the process model and the nature of variations in different process parameters are as follows:

1. A typical cycle of operation of a batch unit consists of steps such as filling, processing, emptying, preparation and/or waiting. For a semicontinuous unit, a cycle of operation consists of steps such as processing and remaining idle.
2. The nominal productivities of both stages are equal, i.e.,

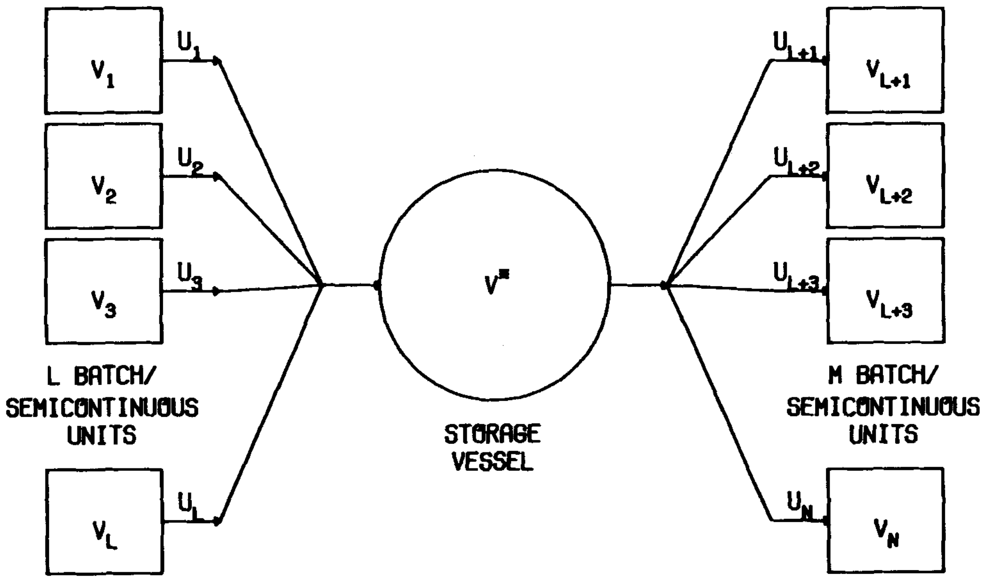

Figure 1. Schematic diagram of a parallel process.

$$\sum_{i=1}^{L} V_i/\omega_i = \sum_{i=L+1}^{N} V_i/\omega_i,$$

where, N is the total number of units, $L + M$.

3. The discharge rates of all the upstream units and the feed rates of all the downstream units, although different, are time-independent and their nominal values are known.

4. The fluctuations arising in various process parameters such as starting moments, flow rates, batch sizes, cycle times, etc., are temporary and nonperiodic in nature. The perturbation in the process parameter introduced at any given time is not permanent and the process returns to its nominal value after the perturbation has occurred. Thus, the process is periodic except for the temporary nonperiodicity introduced by the variations.

5. The batch sizes, the flow rates, and the storage size are expressed in consistent measures (either mass or volume).

6. The required size of the storage tank is equal to the maximum holdup in the tank.

Formulation

In this section, we review the formulation of a general model of the holdup for an L-M process in terms of its nominal parameters.

The nominal cycle time of a batch unit, ω, is given by $\omega = T_f + T_B + T_e + T_p$. The nominal cycle time of a semicontinuous unit is similarly given by, $\omega = T_s + T_i$. For every unit, we define the nominal transfer fraction to be the fraction of the nominal cycle time during which the transfer of material takes place between intermediate storage and the unit. For a batch unit, it will be either an emptying fraction (upstream unit) or a filling fraction (downstream unit) given by,

$$x_i = (T_e)_i/\omega_i \text{ or } X_j = (T_f)_j/\omega_j. \quad (1a,b)$$

For a semicontinuous unit, this will be a processing fraction given by

$$x_i = (T_s)_i/\omega_i. \quad (1c)$$

Let us denote the nominal flow rate for a unit as U_i. For upstream units, U_i is equal to the discharge rate, while for downstream units it is equal to the feed rate. The nominal batch size V_i for the ith unit is defined as the nominal amount of material that it processes in one cycle; hence, V_i is either $U_i(T_e)_i$, or $U_i(T_f)_i$, or $U_i(T_s)_i$, in accordance with the type of unit and the stage in which it is located. Consequently

$$V_i = U_i x_i \omega_i. \quad (2)$$

To each unit i, we assign a nominal transfer flow function, $F_i(t)$, defined as follows,

$$F_i(t) = \begin{cases} c_i U_i & \alpha\omega_i \le t \le (\alpha + x_i)\omega_i \\ 0 & (\alpha + x_i)\omega_i < t < (\alpha + 1)\omega_i \end{cases}$$

where α is an integer and coefficients c_i are defined as follows,

$$c_i = \begin{cases} +1 \text{ if } i\text{th unit is an upstream unit} \\ -1 \text{ if } i\text{th unit is a downstream unit} \end{cases} \quad (3)$$

The starting moment of an upstream unit is defined as the moment when that unit starts discharging material into the storage tank, while the starting moment of a downstream unit is defined as the moment when it starts withdrawing material from the storage tank. Starting moments of units thus do not refer to the start of processing within those units. We arbitrarily select the origin of time as the time beyond which process behavior is to be analyzed. This will usually be the earliest of the starting moments among the N units. The starting moment of the first cycle of the ith unit will be denoted as t_{i0}.

If this process were to operate with parameters fixed at their nominal values and t_{i0} as the starting moments, the holdup in the intermediate storage will be described by,

$$\frac{dV(t)}{dt} = \sum_{i=1}^{N} u(t - t_{i0})F_i(t - t_{i0}) \quad (4)$$

where $u(t)$ is a unit step function. Note that the process described by Eq. 4 is completely periodic only after $t = \max_i (t_{i0})$. We will define a process as periodic if its behavior is described by,

$$\frac{dV(t)}{dt} = \sum_{i=1}^{N} F_i(t - t_{i0}) \quad (5)$$

TYPES OF VARIATIONS

Noncontinuous processes are basically unsteady-state processes and hence the operating policy or schedule has considerable effect on both the design and the operation of these processes. Since such processes are highly labor intensive and often involve complex physical-chemical operations, they are commonly subject to numerous fluctuations in the operating schedule. For ease of analysis, we will classify the various types of fluctuations into two categories: elementary and composite variations. As their names suggest, elementary variations constitute the building blocks for composite variations. The elementary variations may or may not occur by themselves but they are not themselves composed of any other type of variation.

Elementary Variations

There are three types of elementary variations: starting moment revisions, flow rate variations, and transfer fraction variations. We next describe the characteristics of each of these variations.

Starting Moment Revision. Recall that the start of a cycle for an upstream unit is defined as the moment when it starts transferring processed materials to intermediate storage, while the start of a cycle for a downstream unit is defined as the moment when it starts receiving material from storage. For an L-M process with initial starting moments, t_{i0}, $i = 1,N$, the scheduled starting moments of the ith unit, without any variations, will be $t_{i0} + j\omega_i$, $j = 1, 2, \ldots$. A revision of starting moment occurs when a unit does not start its cycle at its scheduled time. Obviously, there are two types of starting moment revisions. If a unit starts earlier than its scheduled starting moment then an advance of starting moment is said to occur, while if a unit starts later than its scheduled starting moment then a delay of starting moment is said to occur. Notice that the entire schedule for a unit is changed as revisions of its starting moments occur. For instance, if we denote the jth revision for the ith unit as Δt_i^j then the schedule for the starting moments of the ith unit after k revisions have taken place is $t_{i0} + \sum_{j=1}^{k} \Delta t_i^j + l\omega_i$, $l = 1, 2, \ldots$. Define

$$t_{ik} = t_{i0} + \sum_{j=1}^{k} \Delta t_i^j, \quad (6)$$

where Δt_i^j is negative for advances and positive for delays. The magnitude of a single advance is limited because a unit can not start earlier than the time at which it emptied itself of the previous batch. In principle, a delay could be of any magnitude.

The revision of starting moment is probably the most likely variation in noncontinuous operation. Often the charge composition or size of a batch varies significantly from cycle to cycle, in which case the processing time required to satisfy product specifications must also be changed. Fluctuations in available utility levels or operator availabilities may also affect the batch processing time, which in turn will manifest itself as a delay or an advance of starting moment of the next cycle. Fluctuations in the emptying and filling times of a unit may also force the revision of a schedule. The

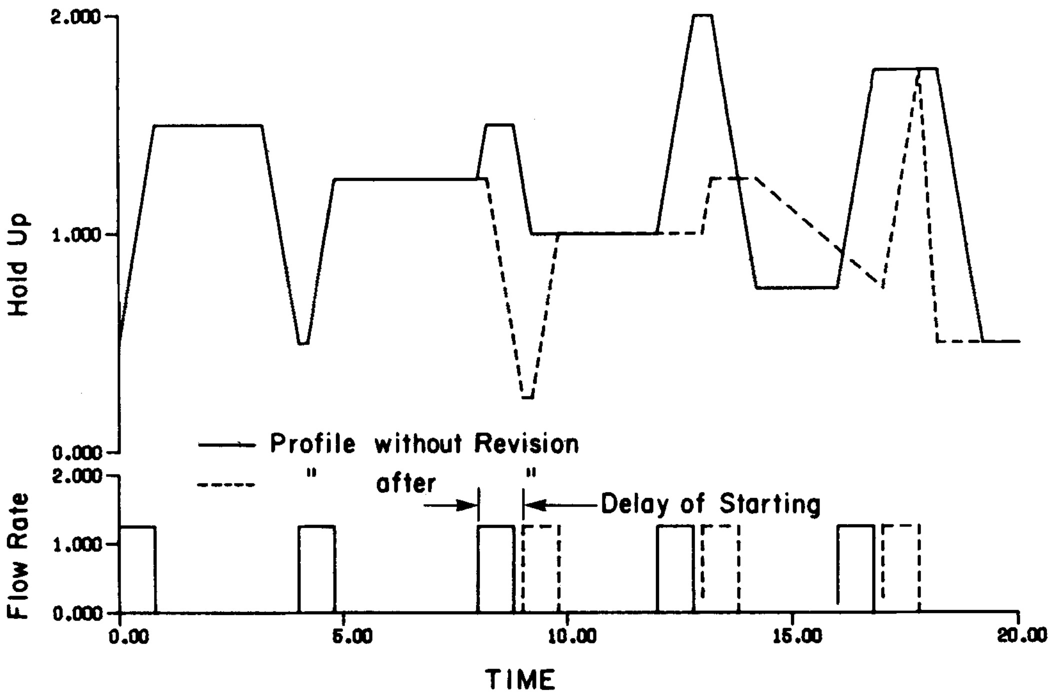

Figure 2. Starting moment revision and its effects.

preparation/cleaning time may also vary significantly depending upon operator availability, the nature and qualities of previous and subsequent batches and many other factors. Moreover, as we shall see subsequently, the starting moment revision is an important constituent of typical composite variations.

Figure 2 illustrates a delay of starting moment for a single unit and the effects on the flow and holdup profiles. Note that all of the scheduled starting moments are changed after the delay occurs. The shape of the holdup profile changes completely for a starting moment revision and the minimum holdup decreases, thereby increasing the initial inventory required for continuity of operation and hence the required size of the intermediate storage vessel.

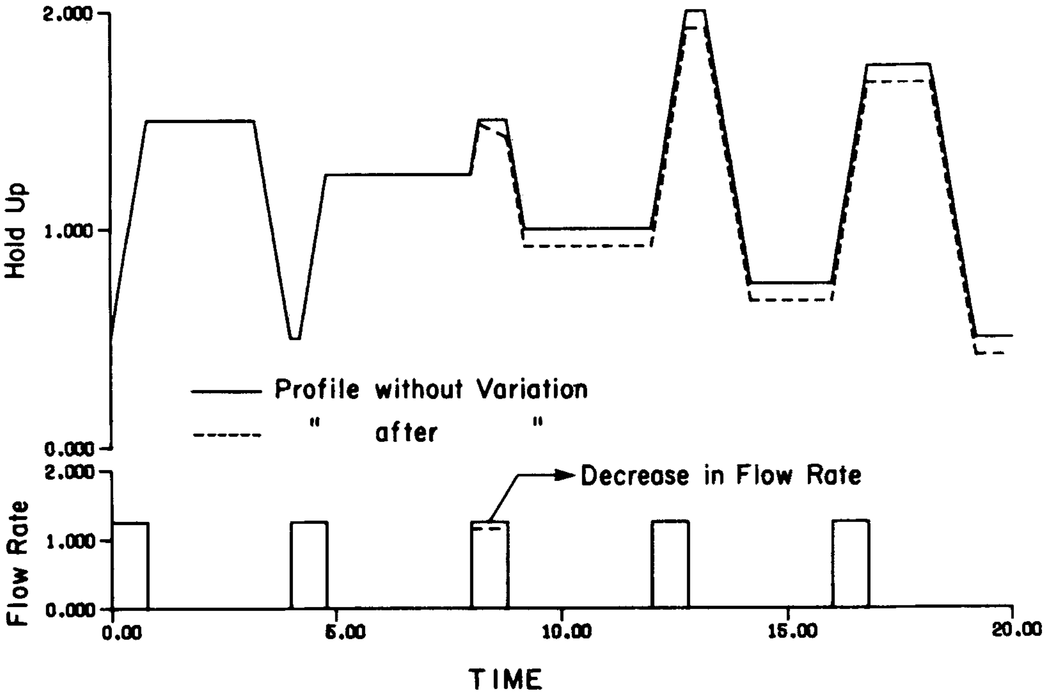

Figure 3. Flow rate variation and its effects.

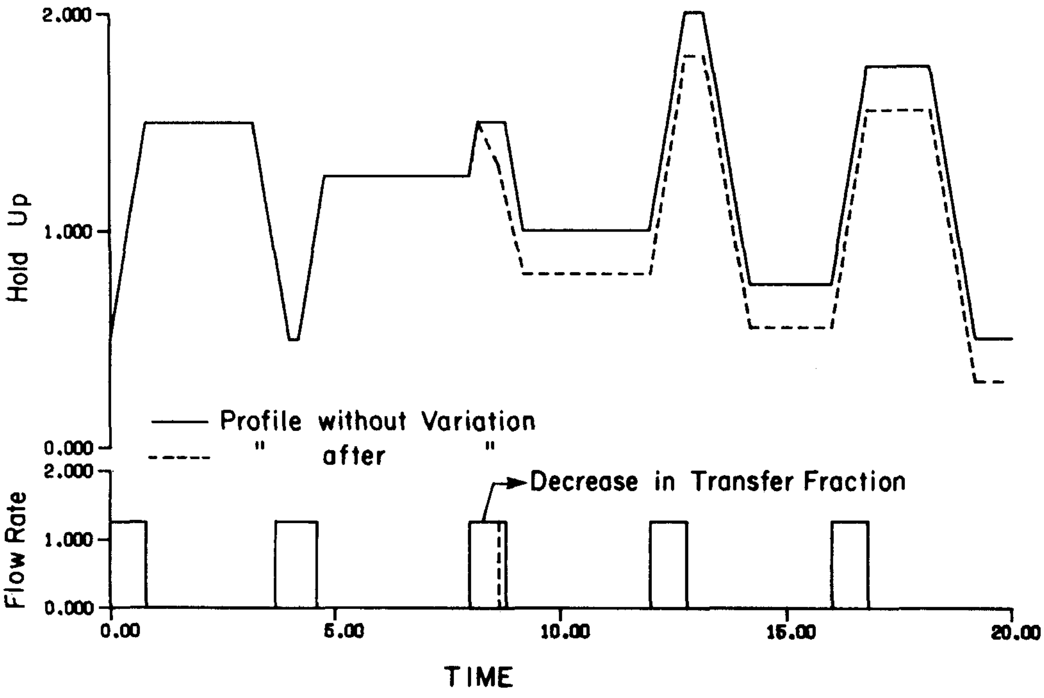

Figure 4. Transfer fraction variation and its effects.

Flow Rate Variation. A flow rate variation occurs in a cycle if the transfer rate between a unit and the intermediate storage deviates from its nominal value during the entire duration of the transfer fraction of the cycle. The process is assumed to return to its nominal values of flow rate in the next cycle unless another variation of this kind occurs. We will denote the jth flow rate variation of the ith unit by ΔU_i^j. The actual flow rate value during this particular variation will be $U_i + \Delta U_i^j$. Notice that the cycle time of the unit and its transfer fraction do not change when an elementary flow rate variation occurs. Operator negligence, pump malfunctioning, or improper utility level may cause a flow rate fluctuation. One may also introduce an intentional flow rate fluctuation in order to reduce the loss of time in transferring an oversized batch.

Figure 3 illustrates a flow rate variation with its effect on the holdup profile. It also illustrates the temporary nature of such a variation as the flow rate assumes its nominal value after the variation has completed in a particular cycle. In contrast to the starting moment revision, here the holdup profile retains its shape but is only shifted uniformly.

Transfer Fraction Variation. A transfer fraction variation occurs if the time required for transferring a batch of material between storage and a unit deviates from its nominal value. Here again the flow rate and cycle time remain unchanged. We will denote the jth transfer fraction variation for the ith unit as Δx_i^j; hence the actual transfer fraction value for this variation will be $x_i + \Delta x_i^j$. We will also assume that the values of the modified transfer fractions always satisfy the constraint

$$0 \leq x_i + \Delta x_i^j < 1 \text{ for all } i,j$$

This is not a very restrictive assumption since the magnitude of such a variation is presumed to be small.

A variation in batch size will normally cause a fluctuation in transfer fraction. However, a fluctuation in flow rate could also cause a change in transfer fraction even for the nominal batch size. A transfer fraction variation may also increase/decrease the cycle time of a unit, particularly if it is a batch unit with negligible preparation and/or waiting time. The nature and effect of a transfer fraction variation is very similar to that of a flow rate variation, as shown in Figure 4.

Variation Intervals. For all elementary variations, we define their intervals of variation as the intervals during which the variation is actually taking place. The starting moment of the ith variation is denoted a_i and the ending moment b_i. The interval for a flow rate variation begins at the start of the cycle and lasts for the length of the transfer fraction. The interval for an increase in the transfer fraction starts at the end of nominal transfer time and lasts until the end of the actual transfer time, while that for a decrease in the transfer fraction starts when the actual transfer time ends, and it ends when the nominal transfer time ends. As far as a delay of starting moment is concerned, the interval starts at the originally scheduled starting moment and ends at the delayed starting moment.

While all the above revisions have a nonzero interval of variation, an advance of a starting moment is most conveniently treated as having a zero interval of variation. One could say that the interval starts at the originally scheduled starting moment and ends at the advanced starting moment. However, since the original flow rate function is zero in this interval, one can assume that at the end of

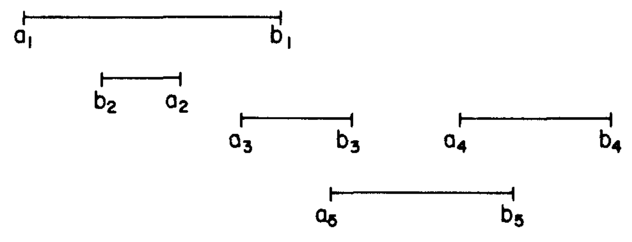

Figure 5. Sequencing of variations.

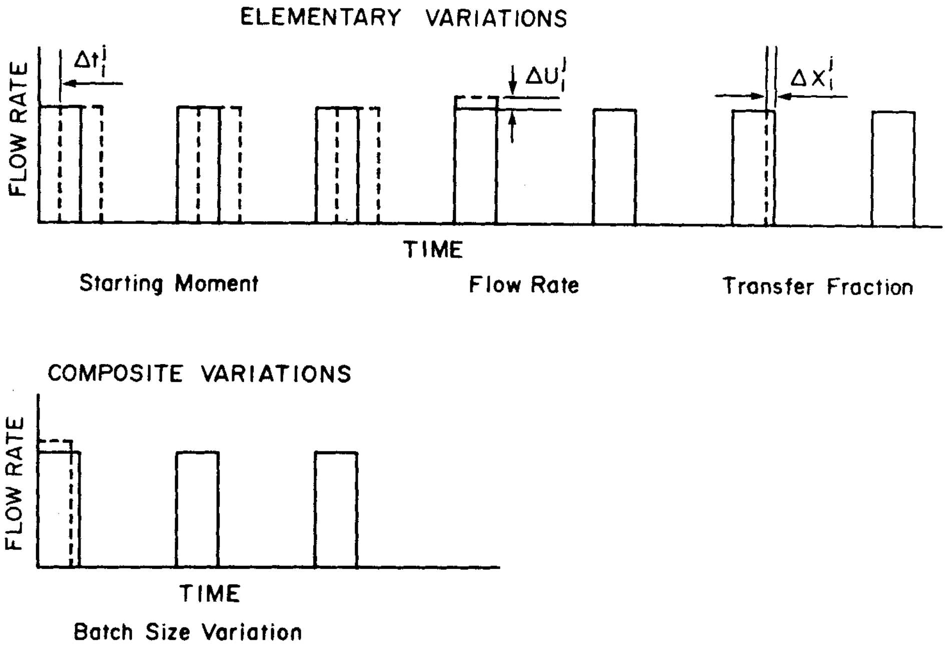

Figure 6. Elementary and composite variations.

an advance in starting moment the original flow rate function instantaneously changes to the revised (advanced) flow rate function. Therefore in our treatment of an advance we will assume its interval of variation to be of zero length.

Next, we will term a pair of variations to be overlapping or nonoverlapping depending upon whether their associated intervals overlap or not. A set of variations will be termed nonoverlapping if no pair of variations in the set has overlapping intervals. By convention, the variations in a set of variations will be sequenced in increasing order of magnitude of the time at which their associated variation intervals end. Thus, the variations in the set of five intervals shown in Figure 5 will be sequenced 2, 1, 3, 5, 4.

The notation $V^i(t)$ will be used to denote the holdup function after the ith variation has completed, assuming that nominal periodic operation continues with no further variation. Thus, $V^i(t)$ represents the holdup for $t \geq b_i$. The function $H_i(t)$ will describe the holdup profile during the variation interval $a_i \leq t \leq b_i$ associated with the ith variation. Finally, the holdup profile for a set of k completed variations will be denoted by $V(k,t)$.

Composite Variations

Usually process variations are more complex than the simple elementary variations discussed in the last section. However, these composite variations can be represented as combinations of elementary variations. There are two common types of composite variations, cycle time variations and batch size variations.

Cycle Time Variation. This is the simplest composite variation. The cycle time of a batch can vary because of changes in any of its constituent step timings. However, we will assume that the transfer fractions and flow rates are at nominal values for a pure cycle time variation. Thus, a pure cycle time variation is merely a revision of starting moment. A decrease in the cycle time results in an advance of the next cycle and vice versa. Any more complex cycle time variation is best treated as a batch size variation.

Batch Size Variation. A batch size variation can arise due to fluctuations in transfer time or transfer flow rate or both. Depending upon whether or not the changes in transfer times are reflected in cycle time changes, there can be two kinds of batch size variations. If there is no change in the cycle time, we have a batch size variation with constant cycle time; otherwise we have a batch size variation with variable cycle time.

A batch size variation with constant cycle time is most likely to occur in the case of a semicontinuous unit. It could also occur in the case of a batch unit with significant idle time, but the treatment remains the same as that for a semicontinuous unit. In a semicontinuous unit, the amount of time during which processing will occur is usually fixed, hence a fluctuation in the processing capacity of the unit will result in a variation in batch size. This is easily treated as an elementary flow variation. Even if the operation of a semicontinuous unit is controlled by amount of material rather than by time, the changes in the processing time affect only the idle time; hence the cycle time can still be assumed to be constant. If the flow rate is at its nominal value, then the variation can be treated as an elementary transfer fraction variation. In principle, one can have both transfer flow rate and transfer fraction variation simultaneously in a cycle. For instance, as shown in Figure 6, if in a cycle starting at $t = a$, the transfer flow rate is U_i' and transfer fraction is x_i', then

$$U_i' = U_i + \Delta U_i^j$$

$$x_i' = x_i + \Delta x_i^k$$

and this batch size variation is a combination of two variations: a flow rate variation during $a \leq t \leq a + \min(x_i,x_i')\omega_i$, followed by a transfer fraction variation during $a + \min(x_i,x_i')\,\omega_i \leq t \leq a + \max(x_j,x_i')\omega_i$. Therefore, any batch size variation with constant cycle time is represented as a sum of two elementary variations: a flow rate variation followed by a transfer fraction variation.

A batch size variation with variable cycle time will only arise in the case of a batch unit with little or no idle time, and constant cleaning/preparation time. Because of the difference in the definitions of starting moments for upstream and downstream units, the required analysis will be slightly different for these two types of units. For instance, as far as the effect on the storage unit is concerned, for an upstream unit, the emptying time is more important than the filling time while for a downstream unit the opposite is true. However, a batch size variation is always introduced while filling a unit and not when emptying it, assuming that the unit is in fact emptied. Since filling precedes the emptying step and the cycle for an upstream unit starts with the emptying step, for such a unit these steps will, for the same batch, occur in different but consecutive cycles. For a downstream unit, on the other hand, the filling and emptying steps for the given batch will both occur in the same cycle.

Let us consider these differences in more detail beginning with the case of an upstream unit. A batch size variation for an upstream unit will only be "visible" when it reaches the emptying step, as the filling step is hidden. Suppose this emptying step occurs in lth cycle of the unit, hence the batch size variation actually originated in the $(l-1)$th cycle. A combination of flow rate and filling time variations could have changed the batch size but only the change in filling time is of importance as far as the intermediate storage is concerned. The lth cycle will undergo a revision of starting moment because of the change in the filling time of the $(l-1)$th cycle. Now since the batch size has changed, a variation in emptying (transfer) flow rate or transfer fraction or both are bound to occur. In general one can consider that this part of the variation is identical to a batch size variation with constant cycle time. However, if the transfer fraction variation occurred then it will introduce a revision of starting moment in $(l+1)$th cycle. Therefore, a batch size variation with variable cycle time can, in the case of an upstream unit, be composed of the following four elementary variations:

1. Starting moment revision at the front of the cycle.
2. Flow rate variation.
3. Transfer fraction variation.
4. Starting moment revision at end of the cycle.

For a downstream unit the composition of batch size variation with variable cycle time is only slighly different. As a cycle of a downstream unit starts with the filling step, there is no starting moment revision at the start of the cycle, but the effect of the change in filling time appears in the starting moment revision at the end of the cycle along with the effect of change in the emptying time. Therefore, for a downstream unit, a batch size variation with variable cycle time can be composed of the following three elementary variations:

1. Flow rate variation.
2. Transfer fraction variation.
3. Starting moment revision at end of the cycle.

Having studied in detail the nature of possible variations in batch/semicontinuous processes, we next proceed to develop sufficiency conditions which these variations must satisfy so that the continuity of operation will not be disturbed. All of the subsequent analysis will be concerned with serial or 1-1 systems. However, specific mention will be made if a particular result extends to systems other than serial ones.

ALLOWABILITY OF VARIATIONS

Before proceeding further with the development, we recall two important definitions first introduced in the work of Oi et al. (1979) and present a preliminary result concerning the initial portion of the holdup function.

Continuability. A holdup function $V(t)$, $t \geq 0$, $V(0) = V_0$, is said to be continuable if continuity of operation can be maintained, that is, for all $t \geq 0$, $0 \leq V(t) \leq V^*$

Allowability. A set of variations in process parameters is allowable, if the corresponding holdup function is continuable.

Starting Holdup. Recall that t_{10} and t_{20} are the starting moments of the up and downstream units in a serial system. Let $\min(t_{10}, t_{20})$ be the time origin (hence, $t_{i0} \geq 0$) and assume that the first variation imposed on the process occurs only after $t = \max(t_{10}, t_{20})$. Furthermore, let V_0 denote the inventory in the storage vessel at $t = 0$ and $V(0,t)$ describe the *starting holdup function*, that is, the holdup function from $t = 0$ to the moment at which the first variation occurs. From Eq. 4 it follows that

$$V(0,t) = V_0 + \sum_{i=1}^{2} \int_0^t u(\tau - t_{i0})F_i(\tau - t_{i0})d\tau$$

and for $t \geq \max(t_{10}, t_{20})$ the holdup is described by the function

$$V^0(t) = V_0 + \sum_{i=1}^{2} \int_{t_{i0}}^t F_i(\tau - t_{i0})d\tau$$

Lemma I. The holdup function $V(0,t)$ is continuable if and only if $V^0(t)$ is continuable.

The proof is given in Appendix I. All of our subsequent results require the continuability of $V(0,t)$. Hence, we will assume throughout that $V^0(t)$ is continuable.

In the next subsections we will derive sufficient conditions to insure the allowability of variations in a 1-1 system. First we will analyze single elementary variations, then sets of multiple elementary variations with and without overlapping intervals, and finally composite variations.

Single Elementary Variations

We consider single elementary variations of each of the three types and derive conditions which each satisfy in order to allow the system to continue with uninterrupted periodic operation.

Starting Moment Revisions. Suppose that a_1 denotes the scheduled starting moment of the jth unit. A starting moment revision occurs at time $t = a_1$ so that the revised starting moment becomes

$$b_1 = a_1 + \Delta t_j^1$$

In case of a delay, i.e. $\Delta t_j^1 > 0$, the holdup profile $V(1,t)$ is given by,

$$V(1,t) = V^0(t) \qquad 0 \leq t \leq a_1$$
$$V(1,t) = H_1(t) \qquad a_1 \leq t \leq b_1$$

and

$$V(1,t) = V^1(t) \qquad t \geq b_1$$

where $V^1(t)$ and $H_1(t)$ are given as follows,

$$V^1(t) = V_0 + \sum_{i=1}^{2} \int_{t_{ik_i}}^t F_i(\tau - t_{ik_i})d\tau$$

$$H_1(t) = V^0(t) - \int_{a_1}^t F_j(\tau - t_{j0})d\tau$$

$$= V^1(t) + \int_t^{b_1} F_j(\tau - t_{j1})d\tau$$

where $k_i = 1$ if $i = j$; otherwise, $k_i = 0$.

Using the above constructions, one can easily prove that for an upstream unit, $V^0(t) \geq H_1(t) \geq V^1(t)$ and for a downstream unit, $V^1(t) \geq H_1(t) \geq V^0(t)$. Since $V^0(t)$ is already assumed to be continuable, the continuability of $V^1(t)$ implies the continuability of $H_1(t)$ and hence that of $V(1,t)$. Thus, if the holdup function after the delay is continuable then the delay is allowable.

In case of an advance, i.e. $\Delta t_j^1 < 0$, $V(1,t)$ is given by,

$$V(1,t) = V_0(t) \qquad 0 \leq t \leq b_1$$
$$V(1,t) = V^1(t) \qquad t \geq b_1$$

Clearly if $V^1(t)$ is continuable, then the advance is allowable. Therefore, a single starting moment revision is allowable if $V^1(t)$ is continuable.

Flow Rate Variation. Let a_1 represent the start of a flow rate variation of amount ΔU_j^1 and $b_1 = a_1 + x_j\omega_j$ represent its end. Then $H_1(t)$ and $V^1(t)$ are given by,

$$H_1(t) = V^0(t) + c_j\Delta U_j^1\,(t - a_1)$$

and

$$V^1(t) = V^0(t) + c_j\Delta U_j^1\, x_j\omega_j$$

Clearly, $V^0(t) \le H_1(t) \le V^1(t)$ if $c_j\Delta U_j^1 \ge 0$ and $V^0(t) \ge H_1(t) \ge V^1(t)$ if $c_j\Delta U_j^1 \le 0$. Therefore, the continuability of $V^1(t)$ implies the allowability of a single flow rate variation.

Transfer Fraction Variation. Let a_1 represent the start of a transfer fraction variation of amount Δx_j^1 and $b_1 = a_1 + \max(x_j,x_j')\omega_j$ represent its end, where $x_j' = x_j + \Delta x_j^1$. Then $H_1(t)$ and $V^1(t)$ are given by,

$$H_1(t) = V^0(t) + c_jU_j(t - a_1)\,\mathrm{sgn}(\Delta x_j^1)$$

and

$$V^1(t) = V^0(t) + c_jU_j\omega_j\Delta x_j^1$$

Again, $V^0(t) \le H_1(t) \le V^1(t)$ if $c_j\Delta x_j^1 \ge 0$, and $V^0(t) \ge H_1(t) \ge V^1(t)$ if $c_j\Delta x_j^1 \le 0$. Therefore, a result similar to those for the other elementary variations is obtained. The above analysis of the three types of variations can be summarized in the following result for single elementary variations.

Proposition I. A single elementary variation is allowable, if the holdup function after the variation is continuable.

The significance of this type of sufficiency condition is that it allows the variation to be characterized without the need to consider the nonperiodic behavior which occurs during the variation interval. It is only necessary to consider the periodic behavior which resumes after the variation transient has terminated. Proposition I is also valid for a general L-M system.

Multiple Elementary Variations

We next develop an allowability condition applicable to any set of multiple elementary variations which occur in a serial system. The set may include any mix of elementary variations and may consist of both overlapping and nonoverlapping intervals.

Theorem I. For a serial system, a set of mixed elementary variations with or without overlaps is allowable if the holdup function after each variation is continuable.

Proof. The proof proceeds by induction. Let ψ_k be a set of k elementary variations having a_i, b_i, $i = 1, k$ as their starting and ending moments, respectively. Clearly from the sequencing convention, $b_1 \le b_2 \le ,,,, \le b_k$. By virtue of proposition I, theorem I is valid for ψ_1. Now, assume that the theorem is valid for ψ_n and form a set ψ_{n+1} by superimposing the $(n + 1)$th variation on ψ_n such that $b_{n+1} \ge b_n$. Let us first assume that the $(n + 1)$th variation is that of a starting moment revision. If the $(n + 1)$th variation is a delay of starting moment, then

$$V(n + 1,t) = V(n,t) \qquad 0 \le t \le a_{n+1}$$
$$V(n + 1,t) = V^{n+1}(t) \qquad t \ge b_{n+1}$$

Now, in a delay of starting moment the transfer flow rate function of the unit undergoing revision must be zero for $a_{n+1} \le t \le b_{n+1}$. Since there are only two units, the net flow function into the intermediate storage must be either nonnegative or nonpositive. Therefore, $V(n + 1,t)$ must be monotonic for $a_{n+1} \le t \le b_{n+1}$. Continuability of $V(n,t)$ implies that $0 \le V(n + 1, a_{n+1}) \le V^*$. If $V^{n+1}(t)$ is continuable then $0 \le V(n + 1, b_{n+1}) \le V^*$ and then the monotonicity of $V(n + 1,t)$ for $a_{n+1} \le t \le b_{n+1}$ will imply the continuability of $V(n + 1,t)$.

If the $(n + 1)$th variation is an advance of starting moment, then

$$V(n + 1,t) = V(n,t) \qquad 0 \le t \le b_{n+1}$$
$$V(n + 1,t) = V^{n+1}(t) \qquad t \ge b_{n+1}$$

Here the continuability of $V^{n+1}(t)$ directly implies the continuability of $V(n + 1,t)$.

Therefore, given that ψ_n is allowable and the holdup function after the $(n + 1)$th variation is continuable, then ψ_{n+1} is allowable. This completes the proof as far as a starting moment revision is concerned. For the other two kinds of elementary variations, the proof proceeds in a similar fashion and is reported in Appendix II.

Note that the above result is quite general since it holds regardless of the mix of types of elementary variations, the magnitude of the variations (combined or individually), or the overlap patterns of the variations. The inductive proof clearly shows that the elementary variations are superimposable and this property will prove quite useful in the subsequent analysis of composite variations. If the variations are nonoverlapping then theorem I holds true for a general L-M system also (Karimi, 1984).

Composite Variations

Recall that a composite variation can be represented as a sum of elementary variations and that the two key composite variations of interest are those in cycle time and those in batch size. As in the case of elementary variations, we first consider single composite variations, followed by a discussion of allowability in the presence of sets of composite variations.

Single Cycle Time Variation. Since a pure cycle time variation is equivalent to a revision of starting moment, the allowability condition for a cycle time variation follows immediately from proposition I. The more complex cycle time variation which entails transfer fraction and flow rate variations will be treated under batch size variations.

Single Batch Size Variation. There are two types of batch size variations of interest: variation with constant cycle time and variation with variable cycle time. We consider each of these in turn.

Suppose a batch size variation with constant cycle time occurs in the lth cycle of operation. Recall that such a variation can be represented as the sum of a flow rate variation and a transfer fraction variation. Let U_j' and x_j' denote the new values of flow rate and transfer fraction in the lth cycle of the jth unit which is undergoing a batch size variation. Denote the resultant batch size as V_j', the batch size variation as ΔV_j^1, and the individual variations as ΔU_j^1 and Δx_j^2. The starting and ending moments of these variations are given by,

$$a_1 = t_{j0} + (l - 1)\omega_j$$
$$b_1 = a_2 = a_1 + \min(x_j, x_j')\omega_j$$
$$b_2 = a_1 + \max(x_j,x_j')\omega_j$$

From the allowability theorem, this composite variation is allowable if $V^1(t)$ and $V^2(t)$ defined below are both continuable.

$$V^1(t) = V^0(t) + c_j\Delta U_j^1\,\omega_j\,\min(x_j,x_j')$$
$$V^2(t) = V^0(t) + c_j\Delta V_j^1$$

Notice that in this composite variation the effect of the first elmentary variation lasts for a short period of time while their combined effect lasts until the next variation. This means that the continuability of $V^2(t)$ is more important than that of $V^1(t)$. It may happen that even if $V^1(t)$ is not continuable, the composite variation may be allowable. Nonetheless, since both $V^1(t)$ and $V^2(t)$ have the same form, one can insure the allowability of the composite by taking the extreme values of the two second terms.

Next, we consider a batch size variation with variable cycle time.

As noted earlier, the treatment of this case is different for an upstream and a downstream unit. Again assume that a batch size variation occurs in the lth cycle. Denote the modified values of flow rate, transfer fraction and batch size in this cycle as U'_j, x'_j and V'_j respectively. The change in the batch size will be denoted by $\Delta V_j^1 = V'_j - V_j$.

Suppose the variation arises with the upstream unit. A batch size variation with variable cycle time can be represented as a series of four elementary variations as follows:

1. Starting moment revision Δt_1^1 for the lth cycle.
2. Flow rate variation Δu_1^2.
3. Transfer fraction variation Δx_1^3.
4. Starting moment revision Δt_1^4 for the $(l + 1)$th cycle.

Their intervals of variations are given by,

$$a_1 = t_{10} + (l - 1)\omega_1$$

$$b_1 = a_2 = a_1 + \Delta t_1^1$$

$$b_2 = a_3 = a_2 + \min(x_1, x'_1)\omega_1$$

$$b_3 = a_2 + \max(x_1, x'_1)\omega_1$$

$$a_4 = b_1 + \omega_1$$

$$b_4 = a_4 + \Delta t_1^4$$

From the allowbility theorem, this composite variation is allowable if $V^1(t)$, $V^2(t)$, $V^3(t)$ and $V^4(t)$ are all continuable. Letting $t_{11} = t_{10} + \Delta t_1^1$, $t_{12} = t_{11} + \Delta t_1^4$, $t_{21} = t_{22} = t_{20}$, we obtain,

$$V^j(t) = V_0 + \sum_{i=1}^{2} \int_{t_{ij}}^{t} F_i(\tau - t_{ij})d\tau \quad j = 0, 1$$

$$V^2(t) = V^1(t) + \omega_1 \Delta U_1^2 \min(x_1, x'_1)$$

$$V^3(t) = V^1(t) + \Delta V_1^1$$

$$V^4(t) = V_0 + \Delta V_1^1 + \sum_{i=1}^{2} \int_{t_{i2}}^{t} F_i(\tau - t_{i2})d\tau$$

Note that $V^0(t)$, $V^1(t)$ and $V^2(t)$, $V^3(t)$ are pairs of holdup functions with similar form. The function $V^4(t)$ modified by adding the second term of $V^2(t)$ appears to be the most general holdup function which can be made allowable by taking contributions from different terms for a worst case analysis.

Next consider a variation arising with a downstream unit. A batch size variation with variable cycle time for a downstream unit can be represented by a series of three elementary variations:

1. Flow rate variation ΔU_2^1.
2. Transfer fraction variation Δx_2^1.
3. Starting moment revision Δt_2^1 for $(l + 1)$th cycle.

The corresponding intervals of variations are given by,

$$a_1 = t_{20} + (l - 1)\omega_2$$

$$b_1 = a_2 = a_1 + \min(x_2, x'_2)\omega_2$$

$$b_2 = a_1 + \max(x_2, x'_2)\omega_2$$

$$a_3 = a_1 + \omega_2$$

$$b_3 = a_3 + \Delta t_2^3$$

Applying the allowability theorem, this composite variation is allowable if $V^1(t)$, $V^2(t)$ and $V^3(t)$ are all continuable. Letting $t_{11} = t_{10}$ and $t_{21} = t_{20} + \Delta t_2^3$, we have,

$$V^1(t) = V^0(t) + \omega_2 \Delta U_2^1 \min(x_2, x'_2)$$

$$V^2(t) = V^0(t) + \Delta V_2^1$$

$$V^3(t) = V_0 + \Delta V_2^1 + \sum_{i=1}^{2} \int_{t_{i1}}^{t} F_i(\tau - t_{i1})d\tau$$

Notice that we get the same general form of holdup function for continuability, as we did for the upstream unit case.

In summary, the allowability conditions for a single composite variation are readily obtained by applying the allowability theorem to the constituent elementary variations. In general, one gets multiple conditions, one for each elementary variation. By considering the specific nature and values of the individual elementary variations, one can easily extract the controlling conditions from the general sets. However, from the point of view of preliminary design, the utility of such a detailed analysis is limited because of the multitude of possible combinations of variations. Therefore, we will limit ourselves to the general sets of conditions which provide the basis for any other simplified sets. In the next section, we will utilize our observation of the general form of holdup function for allowability of single composite variations to analyze multiple variations.

Multiple Composite Variations. The complexity of the analysis of sets of multiple composite variations is evident from the multiple allowability conditions that arise even for single composite variations. Notice that the sequencing of the constituent elementary variations also affects the allowability conditions for single composite variations. A detailed accounting of a set of multiple composite variations is thus impractical because of the large number of combinations of sequences, amounts, and overlaps of variations. However, there are two important results that we can employ in a general treatment of multiple composite variations. First of all, any composite variation can be analyzed as a series of elementary variations. Therefore, a set of multiple composite and/or elementary variations can be treated just as a set of multiple elementary variations with or without overlaps. Secondly, the allowability theorem indicates that the three different types of elementary variations are superimposable. This very important property allows us to treat the overlaps of composite variations as those of elementary ones.

Consider a general set of variations, containing elementary and/or composite variations with no restrictions on their sequence or pattern of overlap. As this set is equivalent to a set of elementary variations, the holdup function after any variation will be comprised of contributions from a certain number of elementary variations. In dealing with this set we do not differentiate between the starting moment revisions arising from diverse sources, e.g., there will be no special treatment for a starting moment revision due to a batch size variation with variable cycle time or for a starting moment revision due to a pure cycle time variation. Furthermore, we assume that a flow rate variation and a transfer fraction variation must both occur if at least one of them occurs in a cycle. Therefore, if a single variation from these two types occurs in a cycle, the amount of variation for the other type will be zero. The resultant of these two variations in a cycle is always a batch size variation with constant cycle time; hence a pair of such variations can be treated as a batch size variation.

Let a set of general variations be comprised of k elementary variations. Furthermore, let k_{ij} be the total number of jth elementary variations for the ith unit, where $j = 1$ denotes starting moment revision, $j = 2$ denotes flow rate variation, and $j = 3$ denotes transfer fraction variation. Clearly, $k = \Sigma_{i=1}^{2} \Sigma_{j=1}^{3} k_{ij}$. The amount of the lth variation of each elementary type for the ith unit will be denoted by Δt_i^l, ΔU_i^l and Δx_i^l, respectively. Let $\Delta V_i^l = (U_i + \Delta U_i^l)(x_i + \Delta x_i^l)\omega_i - V_i$. Then one can formulate a general holdup function as shown in the following proposition (proof given in Appendix III).

Proposition II. For a set of general multiple variations, the holdup function after the kth elementary variation is given by,

$$V^k(t) = V_0 + \sum_{i=1}^{2} \left[c_i \sum_{l=1}^{k_{i3}} \Delta V_i^l + c_i \omega_i (k_{i2} - k_{i3}) \min(x_i, x_i + \Delta x_i^{k_{i2}}) \Delta U_i^{k_{i2}} + \int_{t_{ik_{i1}}}^{t} F_i(\tau - t_{ik_{i1}}) d\tau \right]$$

In the proposition, the terms in brackets represent contributions from the following sources:

1. The first term arises from pairs of elementary variations (a flow rate variation and a transfer fraction variation in the same cycle). In other words, it represents the contriubution from resultant batch size variations.
2. The second term exists only if k is such that there is an incomplete batch size variation (i.e., only a flow rate variation has taken place and the corresponding transfer fraction is still to follow) for any unit. In essence, it is a contribution from a flow rate variation alone.
3. The third term represents a combined effect of starting moment revisions and revised schedule of operation.

Therefore, the allowability of a set of multiple variations of any kind can be expressed in terms of the continuability of the general holdup function described by proposition II for all k.

DISCUSSION

The direct precursor to our results are those reported in the pioneering paper by Oi, et al. (1979). In this section we briefly discuss the differences and similarities in our treatments. Recall from the Scope section, above, that Oi et al. (1979) considered an L-1 system with a downstream continuous unit having a constant processing rate and investigated allowability of single and multiple nonoverlapping starting moment revisions. Specifically, they derived necessary and sufficient conditions for the allowability of these clsses of variations. Since our allowability result (Theorem I) is valid for nonoverlapping variations of all types for general L-M systems, the theorem is a generalization of the sufficiency part of the result of Oi et al. (1979). However, closer examination reveals some important differences. Oi et al. (p. 188) state that the necesary and sufficient condition for the continuability of a system undergoing a set of n nonoverlapping starting moment revisions is that the flow rate function after completion of all revisions, i.e., $V^n(t)$, be continuable. Elaborating further, they state that for a given initial time and V_0 "no order of revisions and magnitude of each revision but the sums of amounts of revisions for each unit determine whether the condition of continuability holds." Thus, the result of Oi et al. only requires continuability after the nth revision, while our condition (theorem I) requires continuability after each revision in the set of n revisions. As the following example indicates, the latter is correct.

Consider a 1-1 system with upstream batch and downstream continuous unit with fixed processing rate. Let $t_{i0} = 0$, $i = 1, 2$; $V_0 = 0$; and $V^* = (U_1 - U_2)x_1\omega_1$. The batch unit has period ω_1; hence, the holdup function has period ω_1. Suppose a delay of starting moment of the batch unit of amount $\epsilon > 0$ occurs at time $j\omega_1$ and then an advance of starting moment of amount $-\epsilon$ occurs at time $\epsilon + (j + 1)\omega_1$. Since the sum of revisions for each unit is zero and $V^2(t) = V(0,t)$ is continuable, the set of two multiple revisions is allowable by the condition of Oi et al. Yet, clearly the set of revisions is not allowable since $V(2,t) = -\epsilon U_2 < 0$ at $t = j\omega_1 + \epsilon$. Our sufficiency test does detect this by requiring continuability after the first revision as well as the second.

The second important difference in our treatments lies in the fact that the necessary and sufficient conditions of Oi et al. require consideration of the continuability of the holdup function after revision both over the transient period during which the variations occur and in the period after the transients have terminated. Our sufficiency conditions are formulated in terms of the periodic holdup functions which arise after the termination of the variations. Thus neither the starting times of the variations nor the overlap patterns of the variations need to be considered. Tightening of our sufficient conditions or, even more so, derivation of any necessary conditions will of necessity require consideration of the holdup function during the transient period. This in turn will require detailed consideration of the start-time of each variation, their overlap patterns, and the sequence in which they occur, in addition to the number, type, and magnitudes of the variations. The complexity of the description of the holdup function which this requires is in our view quite unjustifiable in the context of preliminary design applications. Thus it appears that the derivation of corrected necessary conditions or tighter sufficient conditions is not a fruitful topic for further research. As will be shown in Part II, the generality and simplicity of the allowability conditions presented here do lead directly to convenient and practical sizing procedures for intermediate storage in the presence of all types of parameter variations.

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NOTATION

a_i = starting moment of ith variation
b_i = ending moment of ith variation
c_i = a coefficient assigned to ith unit by Eq. 3
$F_i(t)$ = transfer flow function for ith unit
$H_i(t)$ = holdup function during interval of ith variation
k = number of variations
k_{ij} = number of jth type of variation for ith unit; $j = 1$ starting moment revision, $j = 2$ flow rate variation, $j = 3$ transfer fraction variation
L = number of units in the upstream stage
M = number of units in the downstream stage
N = total number of units, $L + M$
t = time
t_{i0} = initial starting moment of ith unit
t_{ij} = t_{i0} + sum of j starting moment revisions of ith unit as in Eq. 6
T_e = time required to empty a batch unit
T_f = time required to fill a batch unit
T_i = shutdown time for a semicontinuous unit
T_p = preparation time and waiting time for a batch unit
T_B = processing time of a batch unit
T_S = processing time for a semicontinuous unit
$u(t)$ = unit step function
U_i = nominal transfer flow rate of ith unit
U_i' = modified transfer flow rate for ith unit in a cycle
$V(i,t)$ = holdup profile ($t \geq 0$) for i completed variations
$V(t)$ = holdup in the storage vessel
V_0 = initial inventory in storage tank
V_i = nominal batch size of ith unit
V^* = capacity of storage vessel
$V^i(t)$ = holdup function after completion of ith variation without any further variation
x_i = nominal transfer fraction for ith unit as defined by Eq. 1
x_i' = modified transfer fraction for ith unit in a cycle

Greek Letters

α = an integer variable
ψ_k = a set of k elementary variations
Δt_i^j = amount of jth starting moment revision of ith unit
ΔU_i^j = amount of jth flow rate variation of ith unit
ΔV_i^j = amount of jth batch size variation of ith unit
Δx_i^j = amount of jth transfer fraction variation of ith unit

ω_i = nominal cycle time of ith unit
τ = dummy variable

Mathematical Symbols

max[ ] = maximum of the quantities within the brackets
min[ ] = minimum of the quantites within the brackets
sgn(x) = $x/|x|$
| | = absolute value

APPENDIX I: PROOF OF PROPOSITION I

Let us suppose that the jth unit starts earlier than the other unit, denoted as the kth. Therefore, $0 \le t_{j0} \le t_{k0}$, and the holdup profile $V(0,t)$ is given by,

$$V(0,t) = V_0 \quad 0 \le t \le t_{j0}$$

$$V(0,t) = V_0 + \int_{t_{j0}}^{t} F_j(\tau - t_{j0})d\tau \quad t_{j0} \le t \le t_{k0}$$

and

$$V(0,t) = V^0(t) \quad t \ge t_{k0}$$

Since, $0 \le V_0 \le V^*$, $V^0(t)$ is continuable for $0 \le t \le t_{j0}$. Let us assume that if $V^0t)$ is continuable, then $V(0,t)$ is so for $t \ge t_{k0}$. For $j = 1$, i.e., the jth unit is upstream, $F_j(\tau - t_{j0}) \ge 0$, and for $j = 2$, i.e., the jth unit is downstream, $F_j(\tau - t_{j0}) \le 0$; hence $V(0,t)$ is either monotonically increasing ($j = 1$) or monotonically decreasing ($j = 2$) during $t_{j0} \le t \le t_{k0}$. As $V(0,t)$ for $t_{j0} \le t \le t_{k0}$ is monotonic and also bounded by continuable profiles at both ends, $V(0,t)$ is also continuable for $t_{j0} \le t \le t_{k0}$.

Therefore, if $V^0(t)$ is continuable then so is $V(0,t)$. It is obvious that if $V(0,t)$ is to be continuable, so must $V^0(t)$.

APPENDIX II: PROOF OF ALLOWABILITY THEOREM

Flow Rate Variation. Let us assume that the $(n + 1)$th variation is a flow rate variation of amount ΔU_j^{n+1}. Suppose a_{n+1} is the starting moment of this variation and $b_{n+1} = a_{n+1} + x_j\omega_j$ is the ending moment. Then $H_{n+1}(t)$ and $V^{n+1}(t)$ *are given by*

$$H_{n+1}(t) = V(n,t) + c_j\Delta U_j^{n+1}(t - a_{n+1})$$

and

$$V^{n+1}(t) = V(n,t) + c_j\Delta U_j^{n+1}x_j\omega_j$$

Clearly, $V(n,t) \le H_{n+1}(t) \le V^{n+1}(t)$ if $c_j\Delta U_j^{n+1} \ge 0$ and $V(n,t) \ge H_{n+1}(t) \ge V^{n+1}(t)$ if $c_j\Delta U_j^{n+1} \le 0$. Therefore, given the continuability of $V(n,t)$, assuming continuability of $V^{n+1}(t)$ implies the same for $V(n + 1,t)$ for a flow rate variation.

Transfer Fraction Variation. Now let us assume that $(n + 1)$th variation is a transfer fraction variation of amount Δx_j^{n+1}. Therefore, $x_j' = x_j + \Delta x_j^{n+1}$ and $b_{n+1} = a_{n+1} + \max(x_j,x_j')\omega_j$. Then $H_{n+1}(t)$ and $V^{n+1}(t)$ are as follows,

$$H_{n+1}(t) = V(n,t) + c_jU_j(t - a_{n+1})\text{sgn}(\Delta x_j^{n+1})$$

and

$$V^{n+1}(t) = V(n,t) + c_jU_j\omega_j\Delta x_j^{n+1}$$

Again, $V(n,t) \le H_{n+1}(t) \le V^{n+1}(t)$ if $c_j\Delta x_j^{n+1} \ge 0$ and $V(n,t) \ge H_{n+1}(t) \ge V^{n+1}(t)$ if $c_j\Delta x_j^{n+1} \le 0$. Hence, the same result is also valid for a transfer fraction variation.

Thus, we have proved that given the allowability of ψ_n, ψ_{n+1} is also allowable if $V^{n+1}(t)$ is continuable. We already know from Proposition I that given the allowability of ψ_0, ψ_1 is allowable if $V^1(t)$ is continuable. Therefore, by induction, ψ_k is allowable if $V^i(t)$, $i = 0, 1, \ldots k$ are all continuable.

APPENDIX III: PROOF OF PROPOSITION II

We will prove by induction that the holdup function after k elementary variations is given by,

$$V^k(t) = V_0 + \sum_{i=1}^{2}\Bigg[c_1\sum_{l=1}^{k_{i3}} \Delta V_i^l + c_i\omega_i(k_{12} - k_{13})\min(x_i,x_i + \Delta x_i^k)\,\Delta U_i^{k_{12}} + \int_{t_{ik_{i1}}}^{t} F_i(\tau - t_{ik_{i1}})d\tau\Bigg] \quad \text{(A1)}$$

From the treatment of single elementary variations, it is clear that eq. A1 is valid for two cases of $k = 1$. The first case is that of a single starting moment revision for which $k_{i1} = 1$, $k_{i2} = 0$, and $k_{i3} = 0$. The second case is that of a single flow rate variation for which $k_{i1} = 0$, $k_{i2} = 1$, and $k_{i3} = 0$. Notice that a flow rate variation and a transfer fraction variation are assumed to occur as a pair in a cycle, hence $k_{i3} \le k_{i2} \le k_{i3} + 1$ is always satisfied. Therefore, a transfer fraction variation can occur only if $k \ge 2$. For $k = 2$, $k_{i1} = 0$, $k_{i2} = 1$, and $k_{i3} = 1$, Eq. A1 is valid as evident from our treatment of a batch size variation with constant cycle time. Now, let us assume that Eq. A1 is valid for $k = n$ with n_{ij} variations for the ith unit. Let (a_i,b_i) denote the starting and ending moments for the ith variation. Next, we prove that Eq. A1 is valid for $n = n + 1$ when the $(n + 1)$th variation is an elementary one.

Starting Moment Revision. Let the $(n + 1)$th variation be a starting moment revision of the jth unit. This is the $(n_{j1} + 1)$th starting moment revision for the jth unit and also $n_{j2} = n_{j3}$. Since all the variations of a single unit have to be nonoverlapping, we have,

$$V^{n+1}(t) = V^n(t) - \int_{a_{n+1}}^{t} F_j(\tau - t_{jn_{j1}})d\tau + \int_{b_{n+1}}^{t} F_j[\tau - t_{j(n_{j1}+1)}]d\tau$$

For convenience, let us write $V^n(t)$ as,

$$V^n(t) = \phi + \int_{t_{jn_{jl}}}^{t} F_j(\tau - t_{jn_{jl}})d\tau$$

After simplication, we obtain,

$$V^{n+1}(t) = \phi + \int_{t_{j(n_{j1}+1)}}^{t} F_j[\tau - t_{j(n_{jl}+1)}]d\tau$$

Therefore, Eq. A1 is valid for $k = n + 1$ when the $(n + 1)$th variation is a starting moment revision.

Flow Rate Variation. Let the $(n + 1)$th variation be a flow rate variation for the jth unit. From our assumption, a transfer fraction variation must follow this variation, even if it may be of zero amount. For $V^n(t)$, $n_{j2} = n_{j3}$ and the flow rate variation must be the $(n_{j2} + 1)$th variation with amount $\Delta U_j^{n_{j2}+1}$. Similarly, the following transfer fraction must be of amount $\Delta x_j^{n_{j2}+1}$. Then, the holdup function is given by,

$$V^{n+1}(t) = V^n(t) + c_j\omega_j\min(x_j,x_j + \Delta x_j^{n_{j2}+1})\,\Delta U_j^{n_{j2}+1}$$

Here, $k = n + 1$, $k_{j2} = n_{j2} + 1$, and $k_{j3} = n_{j2}$, so Eq. A1 is valid for $k = n + 1$ when the $(n + 1)$th variation is a flow rate variation.

Transfer Fraction Variation. Let the $(n + 1)$th variation be a transfer fraction variation for the jth unit. Clearly, a flow rate variation for the same unit must have preceded this variation for $V^n(t)$; $n_{j2} = n_{j3} + 1$ and this must be the n_{j2}^{th} variation, of amount $\Delta x_j^{n_{j2}}$.

From Eq. A1, $V^n(t)$ can be written as,

$$V^n(t) = \phi + c_j\omega_j \min(x_j, x_j + \Delta x_j^{nj2})\,\Delta U_j^{nj2}$$

then, $V^{n+1}(t)$ is given by,

$$V^{n+1}(t) = \phi + c_j\omega_j[x_j\Delta U_j^{nj2} + (U_j + \Delta U_j^{nj2})\,\Delta x_j^{nj2}] \text{ for } \Delta x_j^{nj2} \geq 0$$

$$V^{n+1}(t) = \phi + c_j\omega_j[(x_j + \Delta x_j^{nj2}, \Delta U_j^{nj2} + U_j\Delta x_j^{nj2}] \text{ for } \Delta x_j^{nj2} < 0$$

Now,

$$\Delta V_j^{nj2} = U_j\omega_j\Delta x_j^{nj2} + \Delta U_j^{nj2}(x_j + \Delta x_j^{nj2})$$

Therefore,

$$V^{n+1}(t) = \phi + c_j\Delta V_j^{nj2}$$

This shows that Eq. A1 is valid for $k = n + 1$ when the $(n + 1)$th variation is a transfer fraction variation. Clearly, Eq. A1 is valid for $k = 1, 2, 3, \ldots$.

LITERATURE CITED

Karimi, I. A., "Analysis of Intermediate Storage in Noncontinuous Processing," Ph.D. Thesis, Purdue Univ. (1984).

Karimi, I. A., and G. V. Reklaitis, "Optimal Selection of Intermediate Storage Tank Capacity in a Periodic Batch/Semicontinuous Process," *AIChE J.*, **29**, 588 (1983).

———, "Intermediate Storage in Noncontinuous Processing," *Proc. 1983 Foundations of Computer-Aided Process Design Conf.*, A. W. Westerberg and H. H. Chien, Eds., 425 (1984).

Oi, K., H. Itoh, and I. Muchi, "Improvement of Operational Flexibility of Batch Units by a Design Margin," *Computers Chem. Eng.*, **3**, 177 (1979).

Oi, K., "Simultaneous Revisions of Schedule of Batch Units with a Continuous Process Permitting a Varying Flow Rate," *Proc. Int. Symp. Process Systems Eng.*, Kyoto, 169–176 (Aug., 1982).

———, "Simultaneous Revision of a Schedule of Batch Units with a Continuous Process Demanding a Constant Flow Rate," *Proc. 3rd Pacific Chem. Eng. Cong.*, Seoul, 15–20 (May, 1983).

Smith, N., and D. Rudd, "On the Effects of Batch Time Variations on Process Performance," *Chem. Eng. Sci.*, **19**, 403 (1964).

Stover, A., "Application of Queueing Theory to Operation and Expansion of a Chemical Plant," *Chem. Eng. Progr. Symp. Ser.*, *42*, **59**, 74 (1963).

Takamatsu, T., et al., "Design of a Flexible Batch Process with Intermediate Storage Tanks," *Ind. Eng. Chem. Process Des. Dev.*, **23**, 40 (1984).

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